

Discrete Voronoi Games and ϵ -Nets, in Two and Three Dimensions

Aritra Banik^{*} Jean-Lou De Carufel[†] Anil Maheshwari^{†‡}

Michiel Smid^{†‡}

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Abstract

The one-round discrete Voronoi game, with respect to a n -point user set U , consists of two players Player 1 (\mathcal{P}_1) and Player 2 (\mathcal{P}_2). At first, \mathcal{P}_1 chooses a set of facilities F_1 following which \mathcal{P}_2 chooses another set of facilities F_2 , disjoint from F_1 . The payoff of \mathcal{P}_2 is defined as the cardinality of the set of points in U which are closer to a facility in F_2 than to every facility in F_1 , and the payoff of \mathcal{P}_1 is the difference between the number of users in U and the payoff of \mathcal{P}_2 . The objective of both the players in the game is to maximize their respective payoffs. In this paper we study the one-round discrete Voronoi game where \mathcal{P}_1 places k facilities and \mathcal{P}_2 places one facility and we have denoted this game as $VG(k, 1)$. Although the optimal solution of this game can be found in polynomial time, the polynomial has a very high degree. In this paper, we focus on achieving approximate solutions to $VG(k, 1)$ with significantly better running times. We provide a constant-factor approximate solution to the optimal strategy of \mathcal{P}_1 in $VG(k, 1)$ by establishing a connection between $VG(k, 1)$ and weak ϵ -nets. To the best of our knowledge, this is the first time that Voronoi games are studied from the point of view of ϵ -nets.

1 Introduction

Facility location is a sub-field of operations research and computational geometry that focuses on the optimal placement of facilities, subject to a set of constraints. One of the most famous facility location problems is the computation of the minimum enclosing disk. Given a set U of n points in the plane, compute the smallest circle that encloses U . There is a $O(n)$ time algorithm by Megiddo [24] to solve this problem as well as a $O(n)$ expected time algorithm by

^{*}Ben-Gurion University of the Negev

[†]Carleton University

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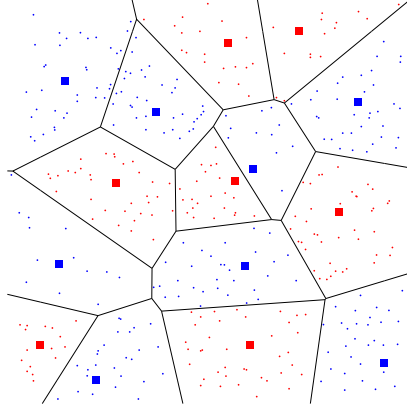


Figure 1: A set U of n of users, denoted by small points, among two competing market players denoted by red and blue squares. When we subdivide the space according to the nearest-neighbour rule, we get the Voronoi diagram of the set of facilities.

Welzl [28]. For an extensive discussion on geometric variants of the facility location problem, refer to Fekete et al. [16].

Competitive facility location (or *competitive spatial modelling*) is concerned with the strategic placement of facilities by competing market players, subject to a set of constraints. In this setting, each facility has its *service zone*, consisting of the set of users it serves. Different metrics can be used to determine the users served by a given facility. In general, the service zone of a player does not have to be connected. In the continuous setting, the objective of each player is to place a set of facilities in order to maximize the total area of its service zone. As for in the discrete setting, the objective of each player is to place a set of facilities in order to maximize the total number of users present in its service zone. The study of competitive facility location started by the work of Hotelling [19] back in 1929. The discrete setting was introduced by Eaton and Lipsey [12]. Numerous variants of facility location problems have been studied (refer to [14, 15, 17] for comprehensive surveys).

Voronoi games, introduced by Ahn et al. [1] (for the one-dimensional case), and by Cheong et al. [10] (for two- and higher-dimensional cases), consist in the following competitive facility location problem. Two players alternately place one facility in \mathbb{R}^d , until each of them has placed a given number of points. Then we subdivide the space according to the nearest-neighbour rule. The player whose facilities control the larger volume wins. In the discrete setting, introduced by Banik et al. [3, 5], the players are given a set U of n users in \mathbb{R}^d (refer to Figure 1). Then, as in the continuous setting, the players alternately place one facility, until each of them has placed a given number of points. Then we subdivide the space according to the nearest-neighbour rule. The player whose service zone contains the largest number of users wins. To *solve* a Voronoi game corresponds to finding an optimal strategy for each player. In this paper, we establish a connection between Voronoi games and weak ε -nets, which leads to general bounds on the scores of the players for discrete Voronoi games.

1.1 Preliminaries and Related Work

In this section, we introduce the notation we use throughout the paper, we define the variants of Voronoi games we study and we explain how these variants compare to existing ones. Consider a set U of n users and let F be a finite set of facilities. In this paper, we identify each facility with its location. For every facility $f \in F$, we define the *service zone of f* , noted $U(f, F)$, as the set of users from U that are served by f . The *discrete Voronoi game* is a competitive facility location problem involving two players \mathcal{P}_1 and \mathcal{P}_2 , respectively. The players \mathcal{P}_1 and \mathcal{P}_2 alternately place two disjoint sets of facilities F_1 and F_2 , respectively. A user $u \in U$ is said to be served by \mathcal{P}_1 (respectively by \mathcal{P}_2) if u belongs to the service zone of at least one facility placed by \mathcal{P}_1 (respectively by \mathcal{P}_2). In such a case, we say that u is in the service zone of \mathcal{P}_1 (respectively of \mathcal{P}_2). The *payoff* $\mathcal{V}(F_1, F_2)$ of \mathcal{P}_2 (or the value of the game) is defined as the cardinality of the set of users from U that belong to the service zone of \mathcal{P}_2 . More formally,

$$\mathcal{V}(F_1, F_2) = \left| \bigcup_{f \in F_2} U(f, F_1 \cup F_2) \right|.$$

Similarly, the payoff of \mathcal{P}_1 is

$$|U| - \mathcal{V}(F_1, F_2) = \left| \bigcup_{f \in F_1} U(f, F_1 \cup F_2) \right|.$$

If $\mathcal{V}(F_1, F_2) > |U| - \mathcal{V}(F_1, F_2)$, we say that \mathcal{P}_2 wins. If $\mathcal{V}(F_1, F_2) < |U| - \mathcal{V}(F_1, F_2)$, we say that \mathcal{P}_1 wins. Otherwise, it is declared a tie. In the *one-round Voronoi Game*, \mathcal{P}_1 places all its facilities after which \mathcal{P}_2 places all its facilities. If $|F_1| = |F_2| = k$, the *k-round Voronoi Game* corresponds to a Voronoi game where the players alternately place their facilities one at a time (refer to [4]).

Let us define the *One-Round Discrete Voronoi Game*.

Definition 1 (One-Round Discrete Voronoi Game $VG(k, l)$). *Let U be a set of n users and \mathcal{P}_1 and \mathcal{P}_2 be two players. Initially, \mathcal{P}_1 chooses a set F_1 of k locations in \mathbb{R}^d for its facilities. Then \mathcal{P}_2 chooses a set F_2 of l locations in \mathbb{R}^d for its facilities, where $F_1 \cap F_2 = \emptyset$.*

1. *Given any choice of F_1 by \mathcal{P}_1 , the objective of \mathcal{P}_2 is to find a set F_2^* of l points, disjoint from F_1 , that maximizes $\mathcal{V}(F_1, F_2)$, where the maximum is taken over all sets of l points $F_2 \subset \mathbb{R}^d$ with $F_1 \cap F_2 = \emptyset$. Formally, the objective of \mathcal{P}_2 is to find a set F_2^* of l points such that*

$$\mathcal{V}(F_1, F_2^*) = \max_{\substack{F_2 \subset \mathbb{R}^d \\ F_1 \cap F_2 = \emptyset \\ |F_2| = l}} \mathcal{V}(F_1, F_2).$$

2. *The objective of \mathcal{P}_1 is to choose a set F_1^* of k facilities to minimize the maximum payoff of \mathcal{P}_2 . In other words, the objective of \mathcal{P}_1 is to find a set F_1^* of k points such that*

$$\max_{\substack{F_2 \subset \mathbb{R}^d \\ F_1 \cap F_2 = \emptyset \\ |F_2| = l}} \mathcal{V}(F_1, F_2)$$

is minimized when $F_1 = F_1^*$, where the minimum is taken of over all sets of k points $F_1 \subset \mathbb{R}^d$ with $F_1 \cap F_2 = \emptyset$.

This game is called the One-Round Discrete Voronoi Game, noted $VG(k, l)$.

In this paper, we study $VG(k, 1)$ for all $k \geq 1$, in two and three dimensions.

Voronoi games were introduced¹ by Ahn et al. [1] (for the one-dimensional case), and by Cheong et al. [10] (for two- and higher-dimensional cases). The discrete version was introduced by Banik et al. [3, 5]. They first studied a version of $VG(k, k)$, where the users in U are restricted to a line. They showed that, if the sorted order of points in U along the line is given, then for any given placement of facilities of \mathcal{P}_1 , an optimal strategy for \mathcal{P}_2 can be computed in linear time. They also provided results for determining an optimal strategy for \mathcal{P}_1 . Then, Banik et al. [4] studied the case where U can be any finite subset of \mathbb{R}^2 . They focused on the following version of the game. The players \mathcal{P}_1 and \mathcal{P}_2 already own two sets of facilities F_1 and F_2 , respectively. The player \mathcal{P}_1 wants to place one more facility knowing that \mathcal{P}_2 will place another facility afterwards. This game is called the *One-Round Discrete Voronoi Game in Presence of Existing Facilities*, or $VG_1^\exists(F_1, F_2)$ for short. The optimal strategy of \mathcal{P}_2 , given any placement of \mathcal{P}_1 , is identical to the solution of the *MaxCov* problem studied by Cabello et al. [6]. Consider a set U of users, two sets of facilities F_1 and F_2 , and any placement of a new facility f by \mathcal{P}_1 . Let $U_1 \subseteq U$ denote the subset of users that are served by \mathcal{P}_1 , in presence of F_1 , F_2 , and f . For every point $u \in U_1$, consider the *nearest facility disk* C_u centered at u and passing through the facility in $F_1 \cup \{f\}$ which is closest to u . Note that a new facility f' placed by \mathcal{P}_2 serves any user $u \in U_1$ if and only if $f' \in C_u$. Let $\mathcal{C} = \{C_u | u \in U_1\}$. Any optimal strategy for \mathcal{P}_2 is a point which is inside a maximum number of circles among the circles in \mathcal{C} . This is the problem of finding the maximum depth in an arrangement of n disks, and can be computed in $O(n^2)$ time [2]. Banik et al. [4] study how this arrangement changes as f and f' move in the plane. They provide a complete characterization of the event points and obtain an algorithm running in $O(n^8)$ time for computing an optimal placement of \mathcal{P}_1 .

Chawla et al. [7, 8] studied the impact of knowing versus not knowing the number of rounds in the game. Suppose that \mathcal{P}_1 does not know in advance the number of rounds, whereas \mathcal{P}_2 does. What is the impact on the payoffs of \mathcal{P}_1 and \mathcal{P}_2 ? Chawla et al. describe how \mathcal{P}_1 suffers from this lack of information. They worked in a framework where the distribution of the users is continuous, but the set of possible locations for the players is finite. They provide bounds on the payoff of \mathcal{P}_1 in that framework.

The game $VG_1^\exists(F_1, F_2)$ where $F_2 = \emptyset$ resembles $VG(k, 1)$. The difference is that in $VG(k, 1)$, \mathcal{P}_1 can choose the location of all of its facilities whereas in $VG_1^\exists(F_1, \emptyset)$, \mathcal{P}_1 can only choose the location of one facility. Moreover, the solution to $VG_1^\exists(F_1, \emptyset)$, takes polynomial time, where the polynomial has a very high degree. In this paper, we focus on achieving approximate solutions to $VG(k, 1)$ with significantly better running times.

We provide a constant-factor approximate solution to the optimal strategy of \mathcal{P}_1 in $VG(k, 1)$. We establish a connection between $VG(k, 1)$ and weak ϵ -nets. To the best of

¹Notice that Shasha [26], in 1992, described a game called the *Territory game*. which corresponds to a variant of a discrete Voronoi game.

our knowledge, this is the first time that Voronoi game is studied from the point of view of ϵ -nets. We propose two different approaches that lead to different bounds (refer to Theorems 6 and 10). We give a bound on the values of k for which the first approach is better than the second one (refer to Proposition 2). As we present the different results, we discuss the algorithmic aspects of the computation of these approximate solutions. We prove that for $k \geq 5$, even though our strategy is approximate, it guarantees that \mathcal{P}_1 wins $VG(k, 1)$ (refer to Corollary 2). These results are presented in Section 2. The first approach is the subject of Subsection 2.3 and the second approach is presented in Subsection 2.4. Analogous results for $VG(k, 1)$ in three dimensions are presented in Section 3. We discuss several open problems in Section 4.

2 Approximate Solutions for $VG(k, 1)$

In this section, we establish a connection between $VG(k, l)$ and weak ϵ -nets. This connection leads to an approximate solution and general bounds on the payoff of \mathcal{P}_1 in $VG(k, 1)$. For the rest of this section, U denotes a set of n users in the plane, in general position. We study the specific cases of $VG(1, 1)$ in Subsection 2.1 and $VG(2, 1)$ in Subsection 2.2. For the general case of $VG(k, 1)$ ($k \geq 1$), we propose two different approaches (both related to weak ϵ -nets). We present them in Subsections 2.3 and 2.4, respectively, and compare them in Subsection 2.4.

2.1 An Approximate Solution for $VG(1, 1)$

Using an approach similar to that of Banik et al. [4], we can solve $VG(1, 1)$ exactly. However, we get a polynomial time algorithm for finding the optimal strategy for \mathcal{P}_1 , where the polynomial has a high degree. This is why in this subsection, as well as in the rest of the paper, we turn to approximation algorithms. The results in this subsection are subsumed in [8]. Nonetheless, we provide proofs for all results since they help understanding the subsequent proofs in the paper.

We start with the following lemma.

Lemma 1. *Let $f_1 \in \mathbb{R}^2$ be the location of the facility placed by \mathcal{P}_1 . There is a strategy for \mathcal{P}_2 that guarantees a payoff of at least $\frac{1}{2}n$ for \mathcal{P}_2 .*

Proof. Let ℓ be any line passing through f_1 . The line ℓ defines two half-planes H_1 and H'_1 , respectively. Without loss of generality, H_1 contains at least half of the users from U . Therefore, by placing one facility $f' \in H_1$ arbitrarily close to f_1 , \mathcal{P}_2 can serve at least $\frac{1}{2}n$ users. \square

In the proof of Lemma 1, we ensure that \mathcal{P}_2 serves at least $\frac{1}{2}n$ users by placing its facility arbitrarily close to f_1 . The argument works because of the general position assumption. The same idea is used at several places in the paper.

A direct consequence of Lemma 1 is that the payoff of \mathcal{P}_1 is at most $\frac{1}{2}n$. To find a lower bound on the payoff of \mathcal{P}_1 , we consider the *centerpoint* (refer to [22, p.14]) of U .

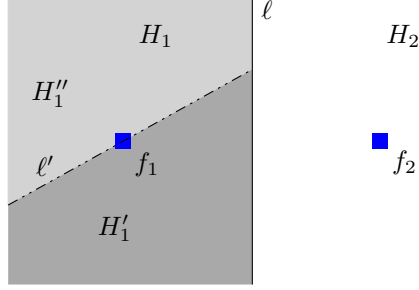


Figure 2: Illustration of the proof of Lemma 3.

Definition 2 (Centerpoint). *Let P be a set of n points in \mathbb{R}^2 . A point $x \in \mathbb{R}^2$ is called a centerpoint of P if each closed half-space containing x contains at least $\frac{1}{3}n$ points of P .*

We have the following theorem about centerpoints (refer to [22, Centerpoint theorem]).

Theorem 1. *Each finite set of points in \mathbb{R}^2 has at least one centerpoint.*

With Theorem 1, we can prove the following lemma.

Lemma 2. *If \mathcal{P}_1 places its facilities at the centerpoint of U , then the payoff of \mathcal{P}_2 is at most $\frac{2}{3}n$.*

Proof. Let f' be the location of the facility of \mathcal{P}_2 . Player \mathcal{P}_2 serves the set of users present in the Voronoi region of f' in the Voronoi diagram of f_1 and f' . However, the Voronoi region of f_1 is a half-plane H_1 which contains f_1 . Therefore, since f_1 is the centerpoint of U , H_1 contains at least $\frac{1}{3}n$ users. Thus, \mathcal{P}_2 can serve at most $\frac{2}{3}n$ users. \square

Therefore,

$$\frac{1}{3}n \leq \text{payoff of } \mathcal{P}_1 \leq \frac{1}{2}n.$$

Consequently, we get a $\frac{n/2}{n/3} = \frac{3}{2}$ -factor approximation of the optimal strategy for \mathcal{P}_1 . Jadhav and Mukhopadhyay [20] showed how to compute the centerpoint of a set of n points in $O(n)$ time, which leads to the following theorem.

Theorem 2. *We can compute in $O(n)$ time a $\frac{3}{2}$ -factor approximation of the optimal strategy for \mathcal{P}_1 in $VG(1, 1)$.*

2.2 An Approximate Solution for $VG(2, 1)$

In this subsection, we present an approximate solution to $VG(2, 1)$. We start with the following lemma.

Lemma 3. *Let $f_1, f_2 \in \mathbb{R}^2$ be the locations of the two facilities placed by \mathcal{P}_1 . There is a strategy for \mathcal{P}_2 that guarantees a payoff of at least $\frac{1}{4}n$ for \mathcal{P}_2 .*

Proof. Let ℓ be the perpendicular bisector of f_1 and f_2 . The line ℓ splits the plane into two half-planes H_1 and H_2 respectively containing f_1 and f_2 (refer to Figure 2). Without loss of generality, suppose that H_1 contains at least $\frac{1}{2}n$ users.

Let ℓ' be any line passing through f_1 . The line ℓ' defines two half-planes H'_1 and H''_1 , respectively. Without loss of generality, H'_1 contains at least half of the users from H_1 . Therefore, by placing one facility $f' \in H'_1$ arbitrarily close to f_1 , \mathcal{P}_2 can serve at least $\frac{1}{4}n$ users. \square

A direct consequence of Lemma 3 is that the payoff of \mathcal{P}_1 is at most $\frac{3}{4}n$. To find a lower bound on the payoff of \mathcal{P}_1 , we use weak ϵ -nets (refer to [22]).

Definition 3 (Weak ϵ -net). *Consider any real number $\epsilon \in [0, 1]$. Let X be a finite set of points in \mathbb{R}^2 and \mathcal{R} be a set of subsets of X . We call the pair (X, \mathcal{R}) a range space. The elements of X and \mathcal{R} are called points and ranges of the range space, respectively.*

Let $N \subseteq \mathbb{R}^2$ be a finite set such that N intersects every set $K \in \mathcal{R}$ with $|K| > \epsilon|X|$. If $N \subseteq X$, then N is an ϵ -net for (X, \mathcal{R}) . Otherwise, N is a weak ϵ -net for (X, \mathcal{R}) .

Mustafa and Ray [25] proved the following theorem.

Theorem 3 (Mustafa and Ray (2009)). *Let P be a set of n points in \mathbb{R}^2 . There exist two distinct points $z_1(P)$ and $z_2(P)$ such that any convex set containing more than $\frac{4}{7}n$ points of P also contains at least one point from $\{z_1(P), z_2(P)\}$.*

With Theorem 3, we can prove the following lemma.

Lemma 4. *If \mathcal{P}_1 places its facilities at $z_1(U)$ and $z_2(U)$, respectively, then the payoff of \mathcal{P}_2 is at most $\frac{4}{7}n$.*

Proof. Suppose that $f_1 = z_1(U)$, $f_2 = z_2(U)$ and \mathcal{P}_2 places its facility at a point f' such that its payoff is more than $\frac{4}{7}n$. Player \mathcal{P}_2 serves the set of users present in the Voronoi region of f' in the Voronoi diagram of f_1, f_2 and f' . However, the Voronoi region of f' is a convex set which does not contain any of f_1 and f_2 but contains more than $\frac{4}{7}n$ points of U . This contradicts Theorem 3. \square

Therefore,

$$\frac{3}{7}n \leq \text{payoff of } \mathcal{P}_1 \leq \frac{3}{4}n.$$

Consequently, we get a $\frac{3n/4}{3n/7} = \frac{7}{4}$ -factor approximation of the optimal strategy for \mathcal{P}_1 . Langerman et al. [21] showed how to compute $z_1(U)$ and $z_2(U)$ in $O(n \log^4(n))$ time, which leads to the following theorem.

Theorem 4. *We can compute in $O(n \log^4(n))$ time a $\frac{7}{4}$ -factor approximation of the optimal strategy for \mathcal{P}_1 in $VG(2, 1)$.*

If we look into the proof of Theorem 3, we see that one of the points in $\{z_1(P), z_2(P)\}$ is a point from P , but in general, the other point is not from P . So in general, $\{z_1(P), z_2(P)\}$ is a weak ϵ -net.

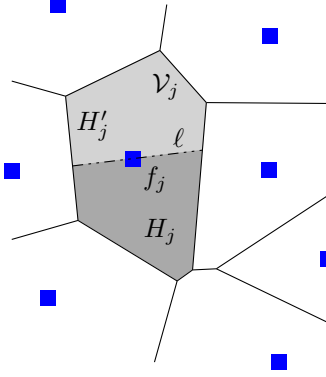


Figure 3: Illustration of the proof of Lemma 5.

2.3 Bounds for $VG(k, 1)$, Where $k \leq 136$

In this section, we study the more general case of $VG(k, 1)$ for $k \geq 2$. We start with the following lemma.

Lemma 5. *Let $\{f_1, f_2, \dots, f_k\} \subset \mathbb{R}^2$ be the set of locations of the k facilities placed by \mathcal{P}_1 . There is a strategy for \mathcal{P}_2 that guarantees a payoff of at least $\frac{1}{2k}n$ for \mathcal{P}_2 .*

Proof. Consider the Voronoi diagram of $\{f_1, f_2, \dots, f_k\}$. It consists in k Voronoi cells (refer to Figure 3). Without loss of generality, suppose that \mathcal{V}_j , the Voronoi cell of f_j , contains at least $\frac{1}{k}n$ users.

Let ℓ be any line passing through f_j . The line ℓ defines two half-planes H_j and H'_j , respectively. Without loss of generality, H_j contains at least half of the users in \mathcal{V}_j . Therefore, by placing one facility $f' \in H_j$ arbitrarily close to f_j , \mathcal{P}_2 can serve at least $\frac{1}{2k}n$ users. \square

A direct consequence of Lemma 5 is that the payoff of \mathcal{P}_1 is at most $\frac{2k-1}{2k}n$. To find a lower bound on the payoff of \mathcal{P}_1 , we use a theorem by Mustafa and Ray [25]. Let us first introduce some notation. Let P be a set of n points in \mathbb{R}^d . Denote by ϵ_i^d the smallest number such that there exists a set N of i distinct points satisfying the following property: any convex set containing more than $\epsilon_i^d n$ points of P also contains at least one point of N .

Given a set \mathcal{K} of convex sets, we say that P pierces \mathcal{K} if for any disk $K \in \mathcal{K}$, $K \cap P \neq \emptyset$. To the best of our knowledge, the only exact values of ϵ_i^d 's that are known are $\epsilon_1^d = \frac{d}{d+1}$ and $\epsilon_2^d = \frac{4}{7}$ (refer to [25, Proposition 3.1]). They are optimal in the following sense. There exist arbitrarily large point sets such that the set of all convex sets containing $\epsilon_i^d n$ points cannot be pierced by i points.

Mustafa and Ray [25, Theorem 2.1] proved the following inequality.

Theorem 5 (Mustafa and Ray (2009)). *Given any integers $r \geq 0$ and $s \geq 0$, we have*

$$\epsilon_{r+ds+1}^d \leq \frac{\epsilon_r^d (1 + (d-1)\epsilon_s^d)}{1 + \epsilon_r^d (1 + (d-1)\epsilon_s^d)}, \quad (1)$$

where $\epsilon_0^d = 1$.

Even though (1) does not lead to the exact values of ϵ_i^d 's in general, we can still use it to find upper bounds on the ϵ_i^d 's. For all integers $d \geq 2$ and $i \geq 0$, we define

$$\bar{\epsilon}_i^d = \begin{cases} 1 & \text{if } i = 0, \\ \min_{\substack{r,s \geq 0 \\ r+ds+1=i}} \frac{\bar{\epsilon}_r^d (1+(d-1)\bar{\epsilon}_s^d)}{1+\bar{\epsilon}_r^d (1+(d-1)\bar{\epsilon}_s^d)} & i > 0. \end{cases} \quad (2)$$

For all $d \geq 2$ and $i \geq 0$, we have $\epsilon_i^d \leq \bar{\epsilon}_i^d$. We can prove it by induction using the fact that if $\epsilon_r^d \leq \bar{\epsilon}_r^d$ and $\epsilon_s^d \leq \bar{\epsilon}_s^d$, then

$$\begin{aligned} \epsilon_i^d &\leq \min_{\substack{r,s \geq 0 \\ r+ds+1=i}} \frac{\epsilon_r^d (1+(d-1)\epsilon_s^d)}{1+\epsilon_r^d (1+(d-1)\epsilon_s^d)} && \text{by (1),} \\ &= \min_{\substack{r,s \geq 0 \\ r+ds+1=i}} \frac{1}{\frac{1}{\epsilon_r^d (1+(d-1)\epsilon_s^d)} + 1} \\ &\leq \min_{\substack{r,s \geq 0 \\ r+ds+1=i}} \frac{1}{\frac{1}{\bar{\epsilon}_r^d (1+(d-1)\bar{\epsilon}_s^d)} + 1} \\ &= \min_{\substack{r,s \geq 0 \\ r+ds+1=i}} \frac{\bar{\epsilon}_r^d (1+(d-1)\bar{\epsilon}_s^d)}{1+\bar{\epsilon}_r^d (1+(d-1)\bar{\epsilon}_s^d)} \\ &= \bar{\epsilon}_i^d && \text{by (2).} \end{aligned}$$

Using a similar argument, we can show that for a fixed d , the sequence $(\bar{\epsilon}_i^d)_{i \geq 0}$ is decreasing.

To prove their theorem (refer to (1)), Mustafa and Ray [25] show the following.

Corollary 1 (Mustafa and Ray (2009)). *Let P be a set of n points in \mathbb{R}^d . There exists a set $E_i^d = \{z_1(P), z_2(P), \dots, z_i(P)\}$ of i distinct points such that any convex set containing more than $\bar{\epsilon}_i^d n$ points of P also contains at least one point of E_i^d .*

We can now give an upper bound on the payoff of \mathcal{P}_2 .

Lemma 6. *If \mathcal{P}_1 places its facilities at all points of E_k^2 , respectively, then the payoff of \mathcal{P}_2 is at most $\bar{\epsilon}_k^2 n$.*

Proof. Suppose that $f_i \equiv z_i(U)$ ($1 \leq i \leq k$) and \mathcal{P}_2 places its facility at a point f' such that its payoff is more than $\bar{\epsilon}_i^d n$. Player \mathcal{P}_2 serves the set of users present in the Voronoi region of f' in the Voronoi diagram of $E_k^2 \cup \{f'\}$. However, the Voronoi region of f' is a convex set which does not contain any point from E_k^2 but contains more than $\bar{\epsilon}_i^d n$ points of U . This contradicts Corollary 1. \square

Therefore,

$$(1 - \bar{\epsilon}_k^2) n \leq \text{payoff of } \mathcal{P}_1 \leq \frac{2k-1}{2k} n. \quad (3)$$

Consequently, we get a $\frac{(2k-1)n/2k}{(1-\epsilon_k^2)n} = \frac{2k-1}{2k(1-\epsilon_k^2)}$ -factor approximation of the optimal strategy for \mathcal{P}_1 . This leads to the following theorem.

Theorem 6. *We can compute in $O(kn \log^4(n))$ time a $\frac{2k-1}{2k(1-\epsilon_k^2)}$ -factor approximation of the optimal strategy for \mathcal{P}_1 in $VG(k, 1)$.*

Proof. We already explained how, by computing E_k^2 , we can get a $\frac{2k-1}{2k(1-\epsilon_k^2)}$ -factor approximation of the optimal strategy for \mathcal{P}_1 in $VG(k, 1)$. It remains to explain how to compute E_k^2 in $O(kn \log^4(n))$ time.

Using dynamic programming, we can compute and store the numbers r_i and s_i ($0 \leq i \leq k$) such that

$$\overline{\epsilon}_i^d = \frac{\overline{\epsilon}_{r_i}^d \left(1 + (d-1)\overline{\epsilon}_{s_i}^d\right)}{1 + \overline{\epsilon}_{r_i}^d \left(1 + (d-1)\overline{\epsilon}_{s_i}^d\right)}$$

(refer to (2)) in $O(k^2)$ time and space.

Let $\mathcal{T}_k(n)$ be the time needed to compute E_k^2 on a set of n points, given the numbers r_i and s_i ($0 \leq i \leq k$). Following the proof of Theorem 2.1 in [25], we can compute E_k^2 by computing the following.

1. Let \mathcal{H} be the set of all half-planes that contain at least $\overline{\epsilon}_i^2$ points of P . For each pair of half-planes $H, H' \in \mathcal{H}$, there is a point $p_{H,H'}$ with minimum y -coordinate. Compute the point with maximum y -coordinate over all $p_{H,H'}$'s. Langerman et al. [21] showed how to compute that “maximin” point in $O(n \log^4(n))$ time.
2. Compute $E_{r_k}^2$ on a set of at most n points, which can be done in at most $\mathcal{T}_{r_k}(n)$ time.
3. Compute $E_{s_k}^2$ on two different sets of at most $\overline{\epsilon}_k^2 n$ points each, which can be done in at most $2\mathcal{T}_{s_k}(\overline{\epsilon}_k^2 n)$ time.

This leads to the following recurrence inequality.

$$\mathcal{T}_k(n) \leq \mathcal{T}_{r_k}(n) + 2\mathcal{T}_{s_k}(\overline{\epsilon}_k^2 n) + O(n \log^4(n)) \quad (4)$$

Using the facts that $k = 1 + r_k + 2s_k$ and $\overline{\epsilon}_i^2 \leq 1$ for all $0 \leq i \leq k$, we can show that (4) solves to $\mathcal{T}_k(n) = O(kn \log^4(n))$ by substitution. Therefore, the total time of computation is

$$O(k^2 + kn \log^4(n)) = O(kn \log^4(n))$$

since $k \leq n$. □

The values of $\frac{2k-1}{2k(1-\epsilon_k^2)}$ are presented in Table 1 for $1 \leq k \leq 10$ and in Figure 4 for $1 \leq k \leq 175$. Even though we do not know the optimal strategy for \mathcal{P}_1 , we can guarantee that it wins when $k \geq 5$.

Corollary 2. *If $k \geq 5$, then \mathcal{P}_1 has a winning strategy for $VG(k, 1)$.*

k	1	2	3	4	5	6	7	8	9	10
$\frac{2k-1}{2k(1-\epsilon_k^2)}$	$\frac{3}{2}$	$\frac{7}{4}$	$\frac{25}{14}$	$\frac{217}{120}$	$\frac{123}{70}$	$\frac{187}{108}$	$\frac{2249}{1302}$	$\frac{1115}{656}$	$\frac{91}{54}$	$\frac{9633}{5740}$
	1.50	1.75	1.79	1.81	1.76	1.73	1.73	1.70	1.69	1.68

Table 1: Values of $\frac{2k-1}{2k(1-\epsilon_k^2)}$ for $1 \leq k \leq 10$ together with numerical values.

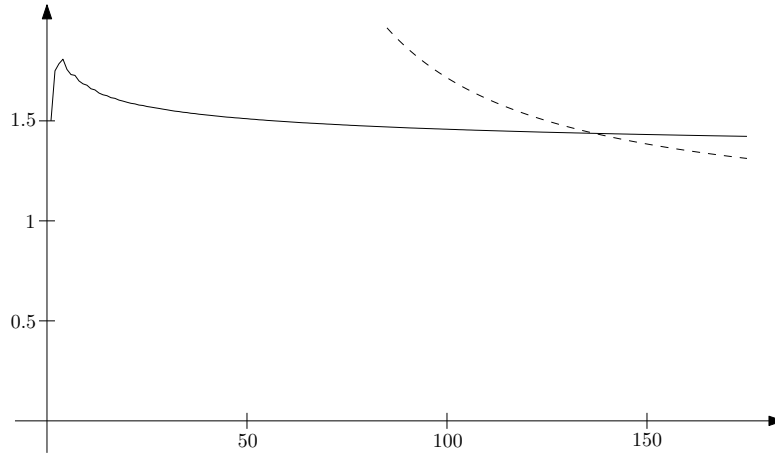


Figure 4: Values of $\frac{2k-1}{2k(1-\epsilon_k^2)}$ for $1 \leq k \leq 175$ (plain curve). Values of $\frac{2k-1}{2(k-42)}$ for $85 \leq k \leq 175$ (dashed curve).

Proof. We have

$$\begin{aligned} \text{payoff of } \mathcal{P}_1 &\geq \left(1 - \overline{\epsilon_k^2}\right) n \\ &> \frac{1}{2}n, \end{aligned}$$

which can be verified numerically for $k \geq 5$ using the fact that the sequence $\left(\overline{\epsilon_i^d}\right)_{i \geq 0}$ is decreasing. \square

In Subsection 2.4, we provide different bounds on the payoff of \mathcal{P}_1 (refer to (7)) using a different approach. The values of the approximation factors from Subsection 2.4 are depicted in Figure 4 for $85 \leq k \leq 175$. We can prove that the strategy of the current subsection is better than the one of Subsection 2.4 for $1 \leq k \leq 136$ (refer to Proposition 2).

2.4 Bounds for $VG(k, 1)$, Where $k \geq 137$

In this section, we obtain a different approximation solution for \mathcal{P}_1 by focusing on the reverse problem. That is, given any $0 < \alpha < 1$, determine whether there exists an integer $\nu(\alpha)$ such that, in $VG(\nu(\alpha), 1)$, there is a placement strategy by \mathcal{P}_1 such that \mathcal{P}_2 can get at most αn users. It is known that for any convex range space and any $0 < \epsilon < 1$, there exists an ϵ -net of size $O\left(\frac{1}{\epsilon} \text{polylog} \frac{1}{\epsilon}\right)$ [9]. Hence, for any real number $0 < \alpha < 1$, $\nu(\alpha) \in O\left(\frac{1}{\alpha} \text{polylog} \frac{1}{\alpha}\right)$. The question is whether $\nu(\alpha) \in O\left(\frac{1}{\alpha}\right)$?

We start with a technical lemma.

Lemma 7. *Let F_1 be the set of facilities placed by \mathcal{P}_1 and f' be the facility placed by \mathcal{P}_2 , such that f' serves at least αn users. There exists a circle which does not contain any of the facilities from F_1 and contains at least $\lceil \frac{\alpha n}{6} \rceil$ users.*

Proof. Denote by $U_{f'}$ the set of at least αn users served by f' . Consider any six rays emerging from f' such that the angle between any two consecutive rays is $\frac{1}{3}\pi$ (refer to Figure 5). These six rays divide the plane into six regions. At least one of these regions contains at least $\frac{\alpha n}{6}$ users from $U_{f'}$. Let λ be such a region and $U_\lambda = U_{f'} \cap \lambda$ be the set of users in λ that are served by f' .

Consider any user $u \in U_\lambda$ which is farthest from f' . Denote the circle centered at f' and passing through u by $C_{f'}$. Let the distance between u and f' be d . Since $u \in U_\lambda$ is farthest from f' , all the users in λ that are served by f' lie in the region $\lambda \cap C_{f'}$. Since the angle between the bounding lines of λ is $\frac{1}{3}\pi$, the maximum distance between any two points in $\lambda \cap C_{f'}$ is d . Hence, the circle C_u centered at u with radius d contains all the users in $\lambda \cap C_{f'}$ that are served by f' . Since u is served by f' , C_u does not contain any facility from F_1 . Hence, the result holds. \square

The following theorem is due to Matoušek et al. [23].

Theorem 7 (Matoušek, Seidel and Welzl (1990)). *Let $0 < \epsilon \leq 1$ be a real number and let \mathcal{D} be a family of disks. For every finite point set S in the plane there exists an ϵ -net with respect to \mathcal{D} of size $O(1/\epsilon)$.*

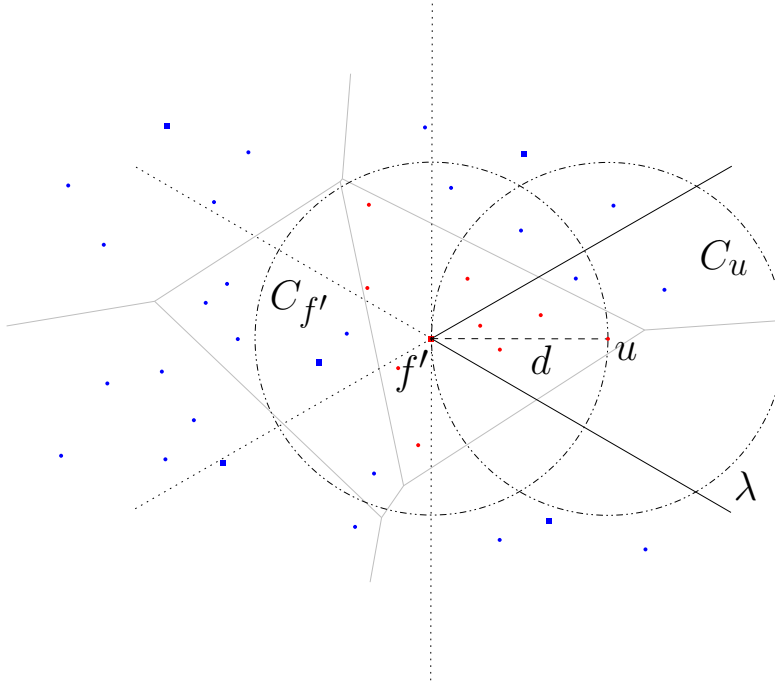


Figure 5: Illustration of the proof of Lemma 7. The Voronoi diagram of $F_1 \cup \{f'\}$ is depicted in grey. The points in red represent the αn users served by \mathcal{P}_2 , i.e. by f' .

We prove the following proposition using Lemma 7 and Theorem 7.

Proposition 1. *For any real number $0 < \alpha < 1$, there exists an integer $k \in O(\frac{1}{\alpha})$ such that in $VG(k, 1)$, \mathcal{P}_1 can choose k points to place its facilities such that the payoff of \mathcal{P}_2 is at most αn .*

Proof. Fix $\epsilon = \frac{\alpha}{6}$. Find an ϵ -net E such that any disk which contains ϵn users contains at least one point from E . By Theorem 7, the size of E is $k \in O(1/\epsilon) = O(1/\alpha)$. We claim that if \mathcal{P}_1 places its facilities at all points of E , \mathcal{P}_2 will get at most αn users. Suppose there exists a placement of facility by \mathcal{P}_2 which serves $\alpha n + 1$ users. From Lemma 7, we know that there exists a disk which does not contain any point from E and contains $\lceil \frac{\alpha n + 1}{6} \rceil > \epsilon n$ users. This contradicts the fact that E is an ϵ -net. \square

From Proposition 1, we know that $\nu(\alpha) \in O(1/\alpha)$. However, the constant hidden in the asymptotic notation is fairly big (refer to [23]). Our next objective is to find a constant κ as small as possible such that $\nu(\alpha) = \kappa/\alpha$.

Let $0 < \epsilon < 1$ be any real number. We know by Theorem 7 that for any $0 < \epsilon \leq 1$, there exists a set of $k \in O(1/\epsilon)$ points which pierces all the disks that contain ϵn points. Next, we prove that given any set P of n points, there exists a set of $7/\epsilon$ points which pierces any disk that contains ϵn points.

Given a set P of n points, let the minimum disk that contains ϵn points from P be D^* . Consider the set \mathcal{D} of all disks D such that $D^* \cap D \neq \emptyset$ and D contains at least ϵn points

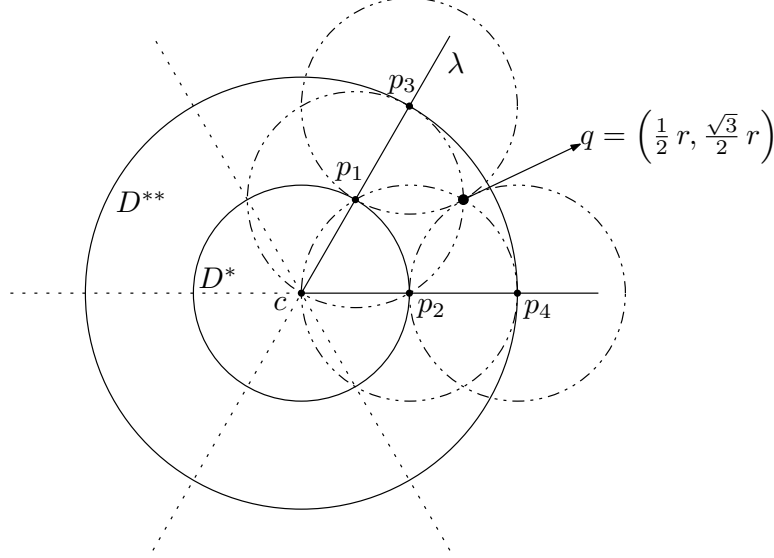


Figure 6: Illustration of the proof of Lemma 8.

from P .

Lemma 8. *The set \mathcal{D} can be pierced by 7 points.*

Proof. Let D^* be centered at c and denote the radius of D^* by r . Consider the disk D^{**} centered at c with radius $2r$. We construct a set Q , which contains 7 points, that pierces \mathcal{D} . Consider any six rays emerging from c such that the angle between any two consecutive rays is $\frac{1}{3}\pi$ (refer to Figure 6). These six rays define six sectors. The set Q consists in one point per sector, plus c .

Let λ be any of the six sectors defined by the six rays. Consider the set $\mathcal{D}_\lambda \subset \mathcal{D}$ of disks that do not contain c and whose centers are in λ . We show that there exists a point that pierces \mathcal{D}_λ .

Observe that the center of any disk in \mathcal{D}_λ must lie outside of D^* because D^* is the minimum-radius disk that contains ϵn points. The disks D^* and D^{**} intersect the boundary of λ in four points p_1 , p_2 , p_3 and p_4 , respectively (refer to Figure 6). Without loss of generality, suppose that $c = (0, 0)$, $p_1 = \left(\frac{1}{2}r, \frac{\sqrt{3}}{2}r\right)$, $p_2 = (r, 0)$ and $p_4 = (2r, 0)$. One can verify that $q = \left(\frac{3}{2}r, \frac{\sqrt{3}}{2}r\right)$ is such that $|p_1q| = |p_2q| = |p_3q| = |p_4q| = r$. We prove that q pierces \mathcal{D}_λ . Let $c_i \in \lambda \setminus D^*$ be the center of a disk D_i in \mathcal{D}_λ . We consider two cases: (1) $c_i \in \lambda \setminus D^{**}$ or (2) $c_i \in \lambda \cap D^{**}$.

1. Without loss of generality, assume that c_i is on the line joining p_2 and p_4 (refer to Figure 7). From the definition of q , we have $|qp_2| = |qp_4| = r$. Furthermore, from the definition of D^{**} , we have $|p_2p_4| = r$. Therefore, the triangle $\triangle qp_2p_4$ is equilateral. Hence, the angle $\angle qp_2p_4 = \frac{1}{3}\pi$. Let $\rho = |p_4c_i|$ and denote the radius of D_i by r_i . Since $D_i \cap D^* \neq \emptyset$, we have

$$r_i \geq r + \rho. \quad (5)$$

Type of Algorithm	Space	Time of Computation	Reference
Deterministic	$O(nk)$	$O(nk \log^2(n))$	[13]
Deterministic	$O(n \log(n))$	$O(nk \log^2(n) \log(n/k))$	[13]
Deterministic	$O(n + k^2)$	$O(n \log(n) + nk)$	[11]
Randomized	$O(n + k^2)$	$O(nk)$	[18]

Table 2: Best known algorithms to compute the minimum k -enclosing disk of a set of n points.

Type of Algorithm	Storage	Time of Computation
Deterministic	$O(\epsilon n^2)$	$O(n^2 \log^2(n))$
Deterministic	$O(n \log(n))$	$O(n^2 \log^2(n) \log(1/\epsilon))$
Deterministic	$O(n + \epsilon^2 n^2)$	$O(\frac{1}{\epsilon} n \log(n) + n^2)$
Randomized	$O(n + \epsilon^2 n^2)$	$O(n^2)$

Table 3: Time of computation for a weak ϵ -net of size $7/\epsilon$.

Proof.

$$\begin{aligned}
\frac{2k-1}{2k(1-\epsilon_k^2)} &\leq \frac{2k-1}{2k(1-\epsilon_k^2)} && \text{refer to the discussion of Section 2.3,} \\
&< \frac{2k-1}{2(k-42)} && \text{can be verified numerically (see Figure 4).}
\end{aligned}$$

□

The running time of the algorithm from the proof of Theorem 8 is related to the computation of the *minimum k -enclosing disk*. That is, given a set P of n points and an integer $1 \leq k \leq n$, find the minimum disk that contains at least k points from P . In Table 2, we present the best known algorithms to solve this problem.

In the algorithm from the proof of Theorem 8, we have $k = \epsilon n$ and we need to compute the minimum k -enclosing disk $1/\epsilon$ times. Therefore, from the algorithms of Table 2, we directly get algorithms to construct the weak ϵ -net of Theorem 8. The running times of these algorithms are presented in Table 3.

3 Voronoi Game in 3 Dimensions

In this section, we study $VG(k, 1)$ in three dimensions. We translate the results and the proofs from Section 2 into three dimensions.

k	1	2	3	4	5	6	7	8	9	10
	2	$\frac{39}{16}$	$\frac{100}{39}$	$\frac{161}{64}$	$\frac{639}{260}$	$\frac{473}{192}$	$\frac{1573}{644}$	$\frac{5505}{2272}$	$\frac{6494}{2691}$	$\frac{22021}{9230}$
	2.00	2.44	2.56	2.52	2.46	2.46	2.44	2.42	2.41	2.39

Table 4: Values of $\frac{2k-1}{2k(1-\epsilon_k^3)}$ for $1 \leq k \leq 10$ together with numerical values.

3.1 $VG(k, 1)$ in 3 Dimensions for $k \leq 805$

This subsection is the three-dimensional version of Subsection 2.3. We start with the following lemma.

Lemma 9. *Let $\{f_1, f_2, \dots, f_k\} \subset \mathbb{R}^3$ be the set of locations of the k facilities placed by \mathcal{P}_1 . There is a strategy for \mathcal{P}_2 that guarantees a payoff of at least $\frac{1}{2k}n$ for \mathcal{P}_2 .*

Proof. The proof is identical to that of Lemma 5 by considering half-spaces instead of half-planes. \square

Then, using Corollary 1, we can prove the following lemma.

Lemma 10. *If \mathcal{P}_1 places its facilities at all points of E_k^3 , respectively, then the payoff of \mathcal{P}_2 is at most $\overline{\epsilon}_k^3 n$.*

Proof. The proof is identical to that of Lemma 6 since three-dimensional Voronoi cells are convex. \square

Therefore,

$$\left(1 - \overline{\epsilon}_k^3\right) n \leq \text{payoff of } \mathcal{P}_1 \leq \frac{2k-1}{2k} n.$$

Consequently, we get a $\frac{(2k-1)n/2k}{(1-\overline{\epsilon}_k^3)n} = \frac{2k-1}{2k(1-\overline{\epsilon}_k^3)}$ -factor approximation of the optimal strategy for \mathcal{P}_1 . The values of $\frac{2k-1}{2k(1-\overline{\epsilon}_k^3)}$ are presented in Table 4 for $1 \leq k \leq 10$, in Figure 9 for $1 \leq k \leq 175$ and in Figure 10) for $1 \leq k \leq 1000$.

In Section 3.2, we provide different bounds on the payoff of \mathcal{P}_1 (refer to (8)) using a different approach. The values of the approximation factors from Subsection 3.2 is depicted in Figure 10 for $660 \leq k \leq 1000$. We can prove that the strategy of the current subsection is better than the one of Subsection 3.2 for $1 \leq k \leq 805$ (refer to Proposition 3).

3.2 $VG(k, 1)$ in 3 Dimensions for $k \geq 806$

Let D be a disk in the plane with radius r and center c . Consider any six rays emerging from c such that the angle between any two consecutive rays is $\frac{1}{3}\pi$. These six rays divide D into six circular sectors. These circular sectors satisfy certain properties that we used in the proofs of Lemmas 7 and 8. In this section, we translate these properties in \mathbb{R}^3 and obtain three dimensional versions of the results of Section 2.4. We start with the following lemma.

Lemma 11. *Let B be a ball in \mathbb{R}^3 with center c . We can cover the whole volume of B using 20 cones with apex c and aperture $\frac{1}{3}\pi$.*

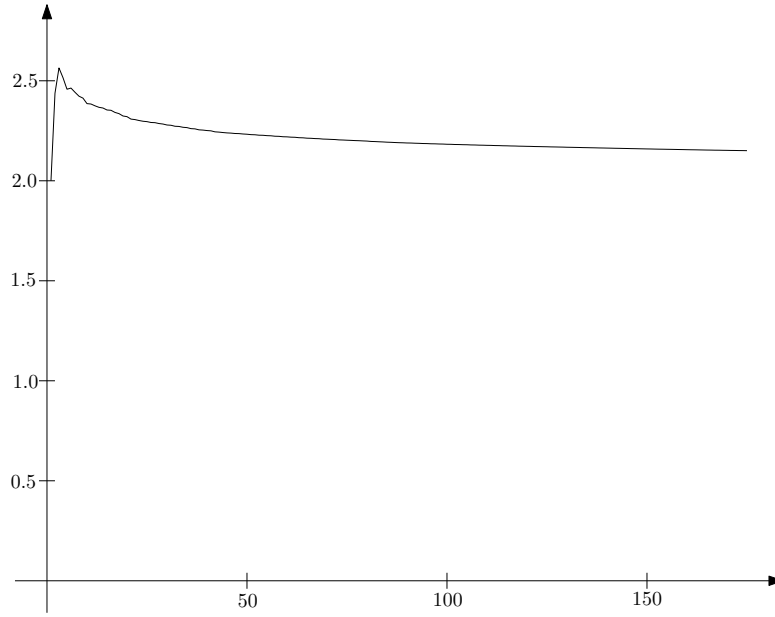


Figure 9: Values of $\frac{2k-1}{2k(1-\epsilon_k^3)}$ for $1 \leq k \leq 175$.

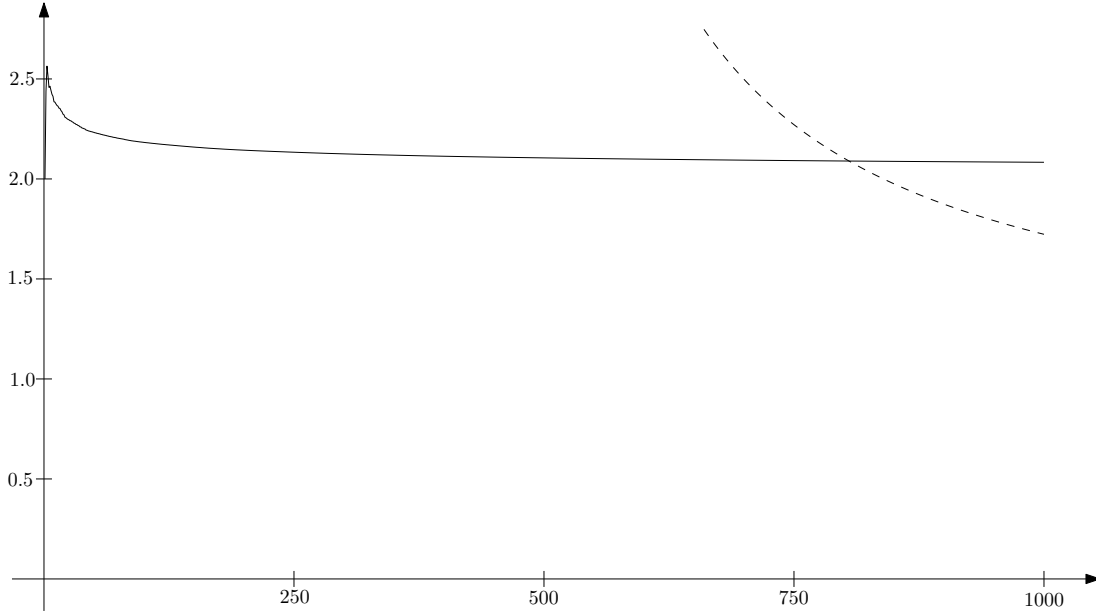


Figure 10: Values of $\frac{2k-1}{2k(1-\epsilon_k^3)}$ for $1 \leq k \leq 1000$ (plain curve). Values of $\frac{2k-1}{2(k-420)}$ for $660 \leq k \leq 1000$ (dashed curve).

Proof. Tarnai and Gáspár [27, Table 2] proved that we can cover the surface of a ball using 20 circles with angular radius less than $0.165\pi < \frac{1}{6}\pi$. Hence, the surface of a ball can be covered using 20 circles with angular radius $\frac{1}{6}\pi$. With each of these circles, we can construct a cone with apex c . Since the circles cover the surface of B , the cones cover B . Moreover, since the angular radius of these circles is $\frac{1}{6}\pi$, the aperture of the cones is $\frac{1}{3}\pi$. \square

Let C be a cone with aperture $\frac{1}{3}\pi$ and radius r . Any ball with radius r and center in C contains C . Together with Lemma 11, this leads to the following lemma.

Lemma 12. *Let F_1 be the set of facilities placed by \mathcal{P}_1 and f' be a facility placed by \mathcal{P}_2 , such that f' serves at least αn users. There exists a ball which does not contain any of the facilities from F_1 and contains at least $\lceil \frac{\alpha n}{20} \rceil$ users.*

Proof. Denote by $U_{f'}$ the set of at least αn users served by f' . Let B be the unit ball centered at f' . Consider the 20 cones from Lemma 11 that cover B . At least one of these cones contains at least $\frac{\alpha n}{20}$ users from $U_{f'}$. Let λ be such a region and $U_\lambda = U_{f'} \cap \lambda$ be the set of users in λ that are served by f' .

Consider any user $u \in U_\lambda$ which is farthest from f' . Denote the ball centered at f' and passing through u by $B_{f'}$. Let the distance between u and f' be d . Since $u \in U_\lambda$ is farthest from f' , all the users in λ that are served by f' lie in the region $\lambda \cap B_{f'}$. Since λ has an aperture of $\frac{1}{3}\pi$, the maximum distance between any two points in $\lambda \cap B_{f'}$ is d . Hence, the ball B_u centered at u with radius d contains all the users in $\lambda \cap B_{f'}$ that are served by f' . Since u is served by f' , B_u does not contain any facility from F_1 . Hence, the result holds. \square

Let $0 < \epsilon < 1$ be any real number. We now explain how to construct a weak ϵ -net Q_ϵ of size $\frac{420}{\epsilon}$ such that any ball which contains ϵn points from P contains at least one point from Q_ϵ . Given a set P of n points, let the minimum ball that contains ϵn points from P be B^* . Consider the set \mathcal{B} of all balls B such that $B^* \cap B \neq \emptyset$ and B contains at least ϵn points from P .

Lemma 13. *The set \mathcal{B} can be pierced by 21 points.*

Proof. Let B^* be centered at c and denote the radius of B^* by r . Consider the ball B^{**} centered at c with radius $2r$. We construct a set Q , which contains 21 points, that pierces \mathcal{B} . Consider any of the 20 cones from Lemma 11 that cover B^* . These 20 cones define 20 sectors. The set Q consists in one point per sector, plus c . Let λ be any of the 20 sectors defined by the 20 cones. Consider the set $\mathcal{B}_\lambda \subset \mathcal{B}$ of balls that do not contain c and whose centers are in λ . We show that there exists a point that pierces \mathcal{B}_λ .

Observe that the center of any ball in \mathcal{B}_λ must lie outside of B^* because B^* is the minimum-radius ball that contains ϵn points. The disks B^* and B^{**} intersect the boundary of λ in two circles C_1 and C_2 , respectively. Without loss of generality, suppose that the boundary of λ satisfies the equation $z = \sqrt{3(x^2 + y^2)}$ (refer to Figure 11). Therefore, C_1 satisfies $x^2 + y^2 = \frac{1}{4}r^2$ and $z = \frac{\sqrt{3}}{2}r$, and C_2 satisfies $x^2 + y^2 = r^2$ and $z = \sqrt{3}r$. One can verify that $q = (0, 0, \sqrt{3}r)$ is such that $|pq| = r$ for all $p \in C_1$ and all $p \in C_2$.

The rest of the proof is similar to that of Lemma 8. \square

From Lemma 13, we can prove that there exists a weak ϵ -net of size $21/\epsilon$.

Proof.

$$\begin{aligned} \frac{2k-1}{2k(1-\epsilon_k^3)} &\leq \frac{2k-1}{2k(1-\epsilon_k^3)} && \text{refer to the discussion of Section 2.3,} \\ &< \frac{2k-1}{2(k-420)} && \text{can be verified numerically (see Figure 10).} \end{aligned}$$

□

The running time of the algorithm from the proof of Theorem 11 is related to the computation of the minimum k -enclosing ball in three dimensions. That is, given a set P of n points and an integer $1 \leq k \leq n$, find the minimum disk that contains at least k points from P . There is an algorithm by Datta et al. [11] to solve that problem in $O(n \log(n) + nk^2 \log^2(k))$ time using $O(n + k^3 \log(k))$ space. In the algorithm from the proof of Theorem 11, we have $k = \epsilon n$ and we need to compute the minimum k -enclosing ball $1/\epsilon$ times. This leads directly to an algorithm to construct the weak ϵ -net of Theorem 11. The running time is $O(\frac{1}{\epsilon} n \log(n) + \epsilon n^3 \log^2(\epsilon n))$ and it uses $O(n + \epsilon^3 n^3 \log(\epsilon n))$ space.

Even though we do not know the optimal strategy for \mathcal{P}_1 , we can guarantee that it wins when $k \geq 841$.

Corollary 3. *If $k \geq 841$, then \mathcal{P}_1 has a winning strategy for $VG(k, 1)$.*

Proof. We have

$$\begin{aligned} \text{payoff of } \mathcal{P}_1 &\geq \frac{k-420}{k}n && \text{by (8),} \\ &> \frac{1}{2}n && \text{since } k \geq 841. \end{aligned}$$

□

4 Conclusion

As we explained in Subsection 1.1, finding an exact solution to $VG(k, 1)$ is a challenging task. By following existing techniques used to solve Voronoi games, we would get polynomial time algorithms to find the optimal strategy for \mathcal{P}_1 , where the polynomial has a very high degree. This is why our goal in this paper was to find approximate solutions with significantly better running times. We studied $VG(k, 1)$ for $k \geq 1$ in two and three dimensions.

In two dimensions, we find two different strategies, both of them based on weak ϵ -nets. The first one (refer to Subsections 2.1, 2.2 and 2.3) was based on weak ϵ -nets for convex sets. More precisely, we used a result by Mustafa and Ray [25] (refer to Corollary 1). The running times and approximation factors we get from that strategy are presented in Table 5. With that approach, we can prove that for $k \geq 5$, \mathcal{P}_1 can win $VG(k, 1)$ with an approximate solution (refer to Corollary 2). The second strategy (refer to Subsection 2.4) was based on weak ϵ -nets for circles. We first proved that there exists weak ϵ -nets for circles of size $7/\epsilon$

k	Approximation Factor	Running time
1	3/2	$O(n)$
2	7/4	$O(n \log^4(n))$
3	25/14	$O(n \log^4(n))$
4	217/120	$O(n \log^4(n))$
5	123/70	$O(n \log^4(n))$
$k > 5$	$\frac{2k-1}{2k(1-\epsilon_k^2)}$	$O(kn \log^4(n))$

Table 5: Running times and approximation factors for $VG(k, 1)$ in two dimensions, where $1 \leq k \leq 5$.

(refer to Theorem 8) and then used that result to approximate the payoff of \mathcal{P}_1 . When $k > 42$, we get a $\frac{2k-1}{2(k-42)}$ -approximation solution (refer to Theorem 10) for which the running time is directly related to the solution of the minimum k -enclosing disk (refer to Table 2). This leads to a polynomial time algorithm, where the polynomial has a low degree. For $VG(k, 1)$ in three dimensions, we get similar results (refer to Section 3). However, we do not know how to compute E_k^3 efficiently. For the two-dimensional case, we could describe a recursive algorithm to compute E_k^2 (refer to Theorem 6). For the three-dimensional case, we leave this question as an open problem.

The game $VG(k, l)$, both from the exact and approximate points of view, leads to numerous open questions. What is the optimal way of solving $VG(k, l)$ exactly (for $l \geq 1$)? What is the connection between $VG(k, l)$ and weak ϵ -nets (for $l > 1$)? When $l = 1$, do we have to compute the minimum k -enclosing disk to build a weak ϵ -net of size $7/\epsilon$ (or $21/\epsilon$ in three dimensions)? If not, can we improve the time of computation of the minimum k -enclosing disk of a set of points? Is there a way of constructing weak ϵ -nets of size smaller than $7/\epsilon$ (or $21/\epsilon$ in three dimensions)? Finally, how tight are our bounds on the payoff of \mathcal{P}_1 ? For instance, are there sets U of users for which \mathcal{P}_1 cannot get more than $(1 - \epsilon_k^2)n$ users (or $(1 - \epsilon_k^3)n$ users in three dimensions)? Are there sets U of users for which \mathcal{P}_1 can achieve a payoff of $\frac{2k-1}{2k}n$? Chawla et al. [7, 8] studied these questions in a different framework. We need to investigate further to see whether their techniques can be applied on our framework.

References

- [1] Hee-Kap Ahn, Siu-Wing Cheng, Otfried Cheong, Mordecai J. Golin, and René van Oostrum. Competitive facility location along a highway. In *COCOON*, pages 237–246, 2001.
- [2] Boris Aronov and Sarel Har-Peled. On approximating the depth and related problems. *SIAM J. Comput.*, 38(3):899–921, 2008.
- [3] Aritra Banik, Bhaswar B. Bhattacharya, and Sandip Das. Optimal strategies for the one-round discrete voronoi game on a line. *J. Comb. Optim.*, 26(4):655–669, 2013.

- [4] Aritra Banik, Bhaswar B. Bhattacharya, Sandip Das, and Satyaki Mukherjee. One-round discrete Voronoi game in \mathbb{R}^2 in presence of existing facilities. In *CCCG*, pages 37–42, 2013.
- [5] Aritra Banik, Bhaswar B. Bhattacharya, and Sandip Das. Optimal strategies for the one-round discrete voronoi game on a line. In Bin Fu and Ding-Zhu Du, editors, *Computing and Combinatorics*, volume 6842 of *Lecture Notes in Computer Science*, pages 213–224. Springer Berlin Heidelberg, 2011.
- [6] Sergio Cabello, José Miguel Díaz-Báñez, Stefan Langerman, Carlos Seara, and Inmaculada Ventura. Facility location problems in the plane based on reverse nearest neighbor queries. *European Journal of Operational Research*, 202(1):99–106, 2010.
- [7] Shuchi Chawla, Uday Rajan, R. Ravi, and Amitabh Sinha. Worst-case payoffs of a location game. In *Proceedings of the 5th ACM Conference on Electronic Commerce, EC '04*, pages 244–245, New York, NY, USA, 2004. ACM.
- [8] Shuchi Chawla, Uday Rajan, R. Ravi, and Amitabh Sinha. Min-max payoffs in a two-player location game. *Oper. Res. Lett.*, 34(5):499–507, 2006.
- [9] Bernard Chazelle, Herbert Edelsbrunner, Michelangelo Grigni, Leonidas J. Guibas, Micha Sharir, and Emo Welzl. Improved bounds on weak epsilon-nets for convex sets. *Discrete & Computational Geometry*, 13:1–15, 1995.
- [10] Otfried Cheong, Sarel Har-Peled, Nathan Linial, and Jiří Matoušek. The one-round voronoi game. In *Proceedings of the Eighteenth Annual Symposium on Computational Geometry, SCG '02*, pages 97–101, New York, NY, USA, 2002. ACM.
- [11] Amitava Datta, Hans-Peter Lenhof, Christian Schwarz, and Michiel H. M. Smid. Static and dynamic algorithms for k -point clustering problems. *J. Algorithms*, 19(3):474–503, 1995.
- [12] B Curtis Eaton and Richard G Lipsey. The principle of minimum differentiation reconsidered: Some new developments in the theory of spatial competition. *The Review of Economic Studies*, pages 27–49, 1975.
- [13] Alon Efrat, Micha Sharir, and Alon Ziv. Computing the smallest k -enclosing circle and related problems. *Comput. Geom.*, 4:119–136, 1994.
- [14] Horst A. Eiselt and Gilbert Laporte. Competitive spatial models. *European Journal of Operational Research*, 39(3):231–242, 1989.
- [15] Horst A. Eiselt, Gilbert Laporte, and Jacques F. Thisse. Competitive location models: A framework and bibliography. *Transportation Science*, 27:44–54, 1993.

- [16] Sándor P. Fekete, Joseph S. B. Mitchell, and Karin Weinbrecht. On the continuous weber and k -median problems (extended abstract). In *Symposium on Computational Geometry*, pages 70–79, 2000.
- [17] Terry L. Friesz, Roger L. Tobin, and Tan Miller. Existence theory for spatially competitive network facility location models. *Ann. Oper. Res.*, 18(1):267–276, 1990.
- [18] Sarel Har-Peled and Soham Mazumdar. Fast algorithms for computing the smallest k -enclosing circle. *Algorithmica*, 41(3):147–157, 2005.
- [19] Harold Hotelling. Stability in competition. *The Economic Journal*, 39(153):41–57, 1929.
- [20] S. Jadhav and A. Mukhopadhyay. Computing a centerpoint of a finite planar set of points in linear time. *Discrete & Computational Geometry*, 12(1):291–312, 1994.
- [21] Stefan Langerman, Mudassir Shabbir, and William Steiger. Computing small hitting sets for convex ranges. In *JCDCGG*, pages 116–117, 2013.
- [22] Jiri Matousek. *Lectures on Discrete Geometry*. Springer-Verlag New York, Inc., Secaucus, NJ, USA, 2002.
- [23] Jiří Matoušek, Raimund Seidel, and E. Welzl. How to net a lot with little: Small epsilon-nets for disks and halfspaces. In *Symposium on Computational Geometry*, pages 16–22, 1990.
- [24] Nimrod Megiddo. Linear-time algorithms for linear programming in \mathbb{R}^3 and related problems. In *FOCS*, pages 329–338, 1982.
- [25] Nabil H. Mustafa and Saurabh Ray. An optimal extension of the centerpoint theorem. *Computational Geometry*, 42(6-7):505 – 510, 2009.
- [26] Dennis Shasha. *Codes, Puzzles, and Conspiracy*. Henry Holt, 1992.
- [27] Tibor Tarnai and Zsolt Gáspár. Covering a sphere by equal circles, and the rigidity of its graph. In *Mathematical Proceedings of the Cambridge Philosophical Society*, volume 110, pages 71–89. Cambridge Univ Press, 1991.
- [28] Emo Welzl. Smallest enclosing disks (balls and ellipsoids). In *Results and New Trends in Computer Science*, pages 359–370. Springer-Verlag, 1991.